ON FINITE DIMENSIONAL LIE ALGEBRAS OF PLANAR VECTOR FIELDS WITH RATIONAL COEFFICIENTS

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Abstract. The Lie algebra of planar vector fields with coefficients from the field of rational functions over an algebraically closed field of characteristic zero is considered. We find all finite-dimensional Lie algebras that can be realized as subalgebras of this algebra.

Introduction

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero and $R = \mathbb{K}(x, y)$ be the field of rational functions. Recall that a $\mathbb{K}$-linear mapping $D : R \to R$ is called a $\mathbb{K}$-derivation if $D(fg) = D(f)g + fD(g)$ for all $f, g \in R$. We denote by $\widetilde{W_2}(\mathbb{K})$ the Lie algebra of all $\mathbb{K}$-derivations of R, this algebra is a two-dimensional vector space over R, its basis $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$ will be called standard. In geometric terms, a derivation D is a vector field with rational coefficients and $\widetilde{W_2}(\mathbb{K})$ is the Lie algebra of all vector fields on $\mathbb{K}^2$ with rational coefficients. The Lie algebra $\widetilde{W_2}(\mathbb{K})$ is closely connected with the automorphism group $\mathrm{Aut}(R)$ of the field R (for example if D is a locally nilpotent derivation of R, then $\exp D$ is an automorphism of R). The group $\mathrm{Aut}(R)$ was intensively studied by many authors (see, for example [3]). A question about finite subgroups of $\mathrm{Aut}(R)$ is of special interest, the description of such subgroups was recently completed by I.Dolgachev and V.Iskovskikh [3]. So, it is of interest to study finite dimensional subalgebras of the Lie algebra $\mathrm{Der}(R) = \widetilde{W_2}(\mathbb{K})$ which corresponds in some sense to $\mathrm{Aut}(R)$.

In this paper, we give a description of finite dimensional subalgebras of $\widetilde{W_2}(\mathbb{K})$ up to isomorphism as Lie algebras using only algebraic tools, so some results can be transferred to Lie algebras of derivations of extensions of fields. Such a description can also be obtained (over the field of complex numbers) using analytical and geometric methods; it can be deduced from results of S.Lie (see [8], S.71-73). There are many papers devoted to such subalgebras, see for example [2], [4], [5], [11], [9], [7], [12]. The main result of the paper is Theorem 1 where all types of finite dimensional subalgebras of $\widetilde{W_2}(\mathbb{K})$ are listed. From this description one can easily obtain all possible types of finite dimensional subalgebras of the Lie algebra $W_2(\mathbb{K}) = \mathrm{Der}\mathbb{K}[x, y]$ (up to isomorphism as Lie algebras). We do not consider the problem of finding all inequivalent realizations of such finite dimensional algebras up to automorphisms of the field $R = \mathbb{K}(x, y)$. Note that this problem in the case $\mathbb{K} = \mathbb{C}$ can be easily solved using results of of S.Lie [8] (see also [4] and [9]).

We use standard notations, the ground field $\mathbb{K}$ is algebraically closed of characteristic zero (some results are valid for any field of characteristic 0). If $D_1, \ldots, D_n$ are elements of

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$\widetilde{W}_2(\mathbb{K})$, then we denote by $\mathbb{K}\langle D_1, \ldots, D_n\rangle$ or simply $\langle D_1, \ldots, D_n\rangle$ the linear span of elements $D_1, \ldots, D_n$ over the field $\mathbb{K}$. The field $\mathbb{K}(x, y)$ of rational functions will be denoted by R, every nonzero $\mathbb{K}$-subspace of $\widetilde{W}_2(\mathbb{K})$ has rank 1 or 2 over R as a system of elements of the two-dimensional vector space $\widetilde{W}_2(\mathbb{K})$ over R.

1. Preliminaries

Lemma 1. *Suppose that $D_1, D_2 \in \widetilde{W}_2(\mathbb{K})$. Then*

(1) *For any $a, b \in R$ it holds $[aD_1, bD_2] = ab[D_1, D_2] + aD_1(b)D_2 - bD_2(a)D_1$.*

(2) *If D_1, D_2 are linearly independent over R and $D_1(c) = D_2(c) = 0$ for some $c \in R$, then $c \in \mathbb{K}$.*

Proof. 1. Straightforward calculation.

2. Note that $\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \in \widetilde{W}_2(\mathbb{K})$ are linear combinations of D_1 and D_2 with coefficients in R. Then $\frac{\partial}{\partial x}(c) = \frac{\partial}{\partial x}(c) = 0$ which implies $c \in \mathbb{K}$. □

Lemma 2. *Let L be a finite dimensional subalgebra of the Lie algebra $\widetilde{W}_2(\mathbb{K})$. If L is of rank 1 over R, then there exists an element $D_1 \in \widetilde{W}_2(\mathbb{K})$ such that L is one of the following algebras:*

(1) *$L = \langle D_1, a_1D_1, \ldots, a_nD_1\rangle$ for some $a_i \in R$ such that $D_1(a_i) = 0$ for all i. The algebra L is abelian.*

(2) *$L = \langle D_1, a_1D_1, \ldots, a_{n-1}D_1, bD_1\rangle$ for some $a_i, b \in R$ such that $D_1(a_i) = 0$ for all i, $D_1(b) = -1$. L is metabelian.*

(3) *$L = \langle D_1, -a^2D_1, -2aD_1\rangle$ for some $a \in R$ with $D_1(a) = 1$. The algebra L is isomorphic to $sl_2(\mathbb{K})$.*

Proof. Replacing the polynomial ring $\mathbb{K}[x, y]$ by the field $R = \mathbb{K}(x, y)$ in the proof of Theorem 1 in [1] one can show that a finite dimensional subalgebra of rank 1 over R from $\widetilde{W}_2(\mathbb{K})$ is either abelian, or metabelian of the form $L = \langle b\rangle \leftthreetimes A, [b, a] = a$ for all $a \in A$ with abelian A, or $L \simeq sl_2(\mathbb{K})$. Consider all these cases. If L is abelian with $\mathbb{K}$-basis $\{D_1, a_1D_1, \ldots, a_nD_1\}$ then $[D_1, a_iD_1] = 0 = D_1(a_i)D_1$ for all i. Hence $D_1(a_i) = 0$ for all i and L is of type 1. Let $L = \langle b\rangle \leftthreetimes A$ with abelian subalgebra $A = \{D_1, a_1D_1, \ldots, a_{n-1}D_1\}$. Then as above $D_1(a_i) = 0$ for all i and since $[bD_1, D_1] = D_1$ we get $D_1(b) = -1$. Thus L is of type 2. Finally, let $L \simeq sl_2(\mathbb{K})$. Choose the standard basis $\{e, f, h\}$ of L over $\mathbb{K}$. Without loss of generality we may put $e = D_1, f = bD_1, h = aD_1$ for some $a, b \in R$. Then

$$[aD_1, D_1] = 2D_1, [aD_1, bD_1] = -2bD_1, [D_1, bD_1] = aD_1,$$

so using Lemma 1 we get from the first equality that $D_1(a) = -2$. The second equality implies $aD_1(b) + 2b = -2b$ and therefore $D_1(b) = -4b/a$. The third equality yields $D_1(b) = a$. So, $a = -4b/a$ and $a^2 = -4b$, i.e. $b = -a^2/4$. We get the basis $\{D_1, -a^2/4D_1, aD_1\}$ of the algebra L. Replacing here a by $-a/2$ we obtain a basis $\{D_1, -a^2D_1, -2aD_1\}$ where $D_1(a) = 1$. □

Remark 1. One can easily point out realizations for Lie algebras from the previous Lemma: 1. $D_1 = \frac{\partial}{\partial x}, a_i = y^i, i = 1, \ldots, n$; 2. $D_1 = \frac{\partial}{\partial x}, a_i = y^i, i = 1, \ldots, n-1, b = -x$; 3. $D_1 = \frac{\partial}{\partial x}, a = x$.

Lemma 3. *Let $L \neq 0$ be a finite dimensional solvable subalgebra of the Lie algebra $\widetilde{W}_2(\mathbb{K})$ and let $\langle D_1 \rangle$ be its arbitrary one-dimensional ideal. Then*
(1) *The set $I = RD_1 \cap L$ is an ideal of L.*
(2) $\dim L/I \leq 2$ *and if* $\dim L/I = 2$, *then the quotient algebra L/I is nonabelian.*
(3) *If* $\dim L \geq 5$, *then the ideal I contains all ideals of rank 1 over R from L.*

Proof. 1. Take any element $D \in L$. Since $\langle D_1 \rangle$ is an ideal of L we have $[D, D_1] = \lambda D_1$ for some $\lambda \in \mathbb{K}$ depending on D. Then for any element $aD_1 \in I$ it holds

$$[D, aD_1] = D(a)D_1 + a[D, D_1] = (D(a) + \lambda a)D_1 \in I.$$

Therefore I is an ideal of L.

2. We can obviously assume that $I \neq L$. Choose a one-dimensional ideal $\langle D_2 + I \rangle$ of the quotient algebra L/I. As $D_2 \notin I$ the elements D_1, D_2 are linearly independent over R. It suffices to show that the ideal $J = I + \langle D_2 \rangle$ of the algebra L is of codimension ≤ 1 in L. Take arbitrary elements $D_3 = a_3D_1 + b_3D_2,\ D_4 = a_4D_1 + b_4D_2$ with $a_3, a_4, b_3, b_4 \in R$ from the set $L \setminus J$. Since

$$[D_1, D_3] = D_1(a_3)D_1 + D_1(b_3)D_2 + b_3\lambda D_1 \in \langle D_1 \rangle$$

(here $[D_2, D_1] = \lambda D_1$) we get $D_1(b_3) = 0$. Analogously from the relation $[D_2, D_3] \in J$ we have $D_2(b_3) = c_3 \in \mathbb{K}$. Similar calculations yield $D_1(b_4) = 0, D_2(b_4) = c_4 \in \mathbb{K}$. It can be easily shown that $c_3 \neq 0, c_4 \neq 0$. Indeed, let to the contrary $c_3 = 0$. Then the equalities $D_1(b_3) = 0, D_2(b_3) = c_3 = 0$ imply by Lemma 1 that $b_3 \in \mathbb{K}$. This means that $a_3D_1 \in L$ and as $a_3D_1 \in I$ we get $D_3 \in J$. The latter contradicts to the choice of D_3. Analogously one can show that $c_4 \neq 0$. Consider the element $c_4D_3 - c_3D_4$ of L and write it in the form

$$(c_4a_3 - c_3a_4)D_1 + (c_4b_3 - c_3b_4)D_2 = r_1D_1 + r_2D_2.$$

Straightforward calculation shows that $D_2(r_2) = 0$. As also $D_1(r_2) = 0$, the element r_2 belongs to $\mathbb{K}$ by Lemma 1. Therefore $c_4D_3 - c_3D_4 \in J$ and D_3, D_4 are linearly dependent over J, i.e. $\dim L/J \leq 1$.

Now let $\dim L/I = 2$ and $\{D_2 + I, aD_1 + bD_2 + I\}$ be a basis of L/I. Suppose that L/I is abelian. Then $[D_2, aD_1 + bD_2] \in I$ and therefore $D_2(b) = 0$. From the relation $[aD_1 + bD_2, D_1] \in \langle D_1 \rangle$ it follows that $D_1(b) = 0$. But then Lemma 1 yields $b \in \mathbb{K}$ which implies $aD_1 \in L$. This means that $aD_1 \in I$ and $aD_1 + bD_2 \in I + \langle D_2 \rangle$. The latter is impossible because the elements D_2 and $aD_1 + bD_2$ are linearly independent over I. This contradiction shows that L/I is nonabelian.

3. Finally, let $\dim L \geq 5$, $I = RD_1 \cap L$ and $T = RD_2 \cap L$ for some ideals $\langle D_1 \rangle$ and $\langle D_2 \rangle$. Suppose that elements D_1 and D_2 linearly independent over R. Since $\dim L/I \leq 2$ and $dim L/T \leq 2$ (by the above proven) and $I \cap T = 0$ we get $\dim L \leq 4$ which contradicts to our assumption. Thus, I contains all ideals of rank 1 over R. □

We need also some elementary properties of rational functions in a single variable. These properties seem to be known but having no reference we supply them with complete proofs. For a rational function $\varphi \in \mathbb{K}(t)$ we will denote $\varphi' = \frac{d\varphi}{dt}$. If $p(t) \in \mathbb{K}[t]$ is an irreducible polynomial, then $\text{ord}_p\varphi$ denotes as usually the integer α from the decomposition of φ into the product of the form $\varphi = p^\alpha\psi$, where neither numerator nor the denominator of ψ is divisible by p.

Lemma 4. *Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. Then:*

(1) *If $\varphi(t) \in \mathbb{K}(t)\backslash\mathbb{K}$, then there does not exist any function $\psi \in \mathbb{K}(t)$ such that $\psi' = \frac{\varphi'}{\varphi}$.*

(2) *Let $\varphi, \psi \in \mathbb{K}(t)\backslash\mathbb{K}$ be such functions that $\mu\varphi'\psi - \varphi\psi' = 0$ for some $\mu \in \mathbb{K}$. Then $\mu \in \mathbb{Q}$, $\mu = \frac{m}{n}$, and $\varphi^m = c\psi^n$ for some $c \in \mathbb{K}$. Moreover, there exists $\theta \in \mathbb{K}(t)$ such that $\varphi = c_1\theta^s$, $\psi = c_2\theta^t$ for some $c_1, c_2 \in \mathbb{K}$, $s, t \in \mathbb{Z}$.*

Proof. 1. Suppose on the contrary that there exists $\psi \in \mathbb{K}(t)$ such that $\psi' = \frac{\varphi'}{\varphi}$. Let $p \in \mathbb{K}[t]$ be an irreducible polynomial such that $\mathrm{ord}_p(\varphi) \neq 0$. Put $\alpha = \mathrm{ord}_p(\varphi)$. Then $\varphi = p^\alpha q$ and $\varphi' = \alpha p^{\alpha-1}p'q + p^\alpha q'$. Therefore

$$\frac{\varphi'}{\varphi} = \frac{\alpha p' p^{\alpha-1} q + p^\alpha q'}{p^\alpha q} = \frac{\alpha q p' + p q'}{pq}.$$

Since $\mathrm{ord}_p(\alpha q p' + pq') = 0$ it holds $\mathrm{ord}_p\left(\frac{\varphi'}{\varphi}\right) = -1$ (note that $\mathrm{ord}_p(q) = 0$). Now put $\beta = \mathrm{ord}_p(\psi)$, $\psi = p^\beta r$. Then $\psi' = \beta p' p^{\beta-1} r + p^\beta r'$. If $\beta = 0$, then $\psi' = r'$ and $\mathrm{ord}_p(\psi') = \mathrm{ord}_p(r') \geq 0$. Suppose that $\beta \neq 0$. Then

$$\mathrm{ord}_p(\psi') = \mathrm{ord}_p(\beta p' p^{\beta-1} r + p^\beta r') = \mathrm{ord}_p(\beta p' p^{\beta-1} r) = \beta - 1.$$

Therefore in any case $\mathrm{ord}_p \psi' \neq -1$, which contradicts to the equality $\mathrm{ord}_p\left(\frac{\varphi'}{\varphi}\right) = -1$. Hence there does not exist such a polynomial ψ that $\psi' = \frac{\varphi'}{\varphi}$.

2. Take any functions φ, ψ from $\mathbb{K}(t)\backslash\mathbb{K}$ satisfying the condition

$$\mu\varphi'\psi - \varphi\psi' = 0. \tag{1}$$

It can be easily shown that there exists a point $t_0 \in \mathbb{K}$ such that $\mathrm{ord}_{t-t_0}\varphi \neq 0$ (because the field $\mathbb{K}$ is algebraically closed). Without loss of generality we can assume that the field $\mathbb{K}(t)$ is embedded to the field $\mathbb{K}((t))$ of Laurent series at the point t_0. Put

$$\varphi = \sum_{i=m}^{\infty} \alpha_i (t - t_0)^i, \psi = \sum_{i=n}^{\infty} \beta_i (t - t_0)^i, \text{where } m, n \in \mathbb{Z}, \alpha_m \beta_n \neq 0.$$

Since $ord_{t-t_0}\varphi \neq 0$, it holds $m \neq 0$, We can assume that $\alpha_m = \beta_n = 1$, because the equation (1) is homogeneous. Computing coefficients at t^{m+n-1} in both sides of the equation (1) we obtain $\mu m = n$. Therefore $\mu = n/m \in \mathbb{Q}$. Further,

$$\left(\frac{\varphi^n}{\psi^m}\right)' = \frac{n\varphi^{n-1}\varphi'\psi^m - m\varphi^n\psi^{m-1}\psi'}{\psi^{2m}} = \frac{\varphi^{n-1}\psi^{m-1}(n\varphi'\psi - m\varphi\psi')}{\psi^{2m}} = 0,$$

because $n\varphi'\psi - m\varphi\psi' = m(\mu\varphi'\psi - \varphi\psi') = 0$. Hence, $\frac{\varphi^n}{\psi^m} \in \mathbb{K}$ i.e. $\varphi^n = c\psi^m$ for some $c \in \mathbb{K}$.

The functions φ and ψ can be written as products of irreducible factors with (nonzero) integer powers

$$\varphi = \prod_{i=1}^{s} u_i^{k_i}, \ \psi = \prod_{j=1}^{k} v_i^{l_i}.$$

Using the equality $\varphi^n = c\psi^m$ we get $k = s$ and after renumbering the factors we can write down $u_i = \gamma_i v_i$ for some $\gamma_i \in \mathbb{K}$. Hence we have:

$$\left(\prod_{i=1}^{k} u_i^{k_i}\right)^n = c\left(\prod_{i=1}^{k} (\gamma_i u_i)^{l_i}\right)^m.$$

This equality implies that $nk_i = ml_i$ for all $i = 1, \ldots, k$. Denote $d = \gcd(m, n)$ and $m = m_1 d$, $n = n_1 d$. We obtain equalities $n_1 d k_i = m_1 d l_i$, $i = 1, \ldots, k$, and therefore $n_1 k_i = m_1 l_i$. Since $\gcd(m_1, n_1) = 1$ we obtain that l_i is divisible by n_1, k_i is divisible by m_1, $i = 1, \ldots, k$. Denote $\frac{l_i}{n_1} = \frac{k_i}{m_1} = r_i$ and $\theta = \prod_{i=1}^{s} u_i^{r_i}$. Then $\varphi = \theta^{m_1}$, $c_1\psi = \theta^{n_1}$ for some $c_1 \in \mathbb{K}^*$. This completes the proof of Lemma. □

Lemma 5. *Let D_1 and D_2 be elements of $\widetilde{W_2}(\mathbb{K})$ linearly independent over R such that $[D_2, D_1] = \nu D_1$ for some $\nu \in \mathbb{K}$. Let b_1, b_2 be linearly independent over $\mathbb{K}$ elements of $R\backslash\mathbb{K}$ such that $D_1(b_i) = 0, i = 1, 2$. Then:*

(1) *If $[D_2, b_i D_1] = \lambda_i b_i D_1$ for some $\lambda_i \in \mathbb{K}, i = 1, 2$, then $\lambda_1 \neq \lambda_2$. If $\lambda_1 \neq \nu$, then $\frac{\lambda_2 - \nu}{\lambda_1 - \nu} \in \mathbb{Q}$.*

(2) *If $[D_2, b_1 D_1] = \lambda b_1 D_1, [D_2, b_2 D_1] = \lambda b_2 D_1 + b_1 D_1$ for some $\lambda \in \mathbb{K}$, then $\lambda = \nu$.*

Proof. 1. Using the condition $[D_2, b_i D_1] = \lambda_i b_i D_1$ we get

$$D_2(b_i) = (\lambda_i - \nu) b_i, i = 1, 2. \tag{2}$$

Suppose that $\lambda_1 = \lambda_2$. Then $D_2\left(\frac{b_1}{b_2}\right) = \frac{D_2(b_1)b_2 - b_1 D_2(b_2)}{b_2^2} = 0$. Besides, $D_1(\frac{b_1}{b_2}) = 0$ by conditions of Lemma. Then using linear independence of elements D_1, D_2 we obtain by Lemma 1 the inclusion $\frac{b_1}{b_2} \in \mathbb{K}$. The latter is impossible because of linear independence of elements b_1, b_2 over $\mathbb{K}$. Hence $\lambda_1 \neq \lambda_2$.

Let now $\lambda_1 \neq \nu$. Since $b_1, b_2 \in R\backslash\mathbb{K}$, the subfield $\ker(D_1)$ of R is of transcendence degree 1 over $\mathbb{K}$ (it is obvious that this degree cannot be equal to 2). Hence $\ker D_1$ is generated by a single element (see, for example, [13], Th.3 and [10]). Denote this element by θ. Then $b_1 = \varphi_1(\theta)$, $b_2 = \varphi_2(\theta)$ for some rational functions $\varphi_1(t)$, $\varphi_2(t) \in \mathbb{K}(t)$. Using the relation $[D_2, D_1] = \nu D_1$ we see that $D_2(\theta) \in \ker(D_1)$. Denote also $D_2(\theta) = f(\theta)$, $f \in \mathbb{K}(t)$. The conditions (2) imply

$$\varphi_1'(\theta) f(\theta) = (\lambda_1 - \nu)\varphi_1(t), \ \varphi_2'(\theta) f(\theta) = (\lambda_2 - \nu)\varphi_2(\theta).$$

Since φ_i are not constants and $\lambda_1 - \nu \neq 0$ we have:

$$\varphi_1 \varphi_2' - \mu \varphi_1' \varphi_2 = 0, \text{ where } \mu = \frac{\lambda_2 - \nu}{\lambda_1 - \nu}.$$

Now Lemma 4 yields the inclusion $\mu \in \mathbb{Q}$.

2. By the condition (2) of Lemma we have

$$D_2(b_1) = (\lambda - \nu) b_1, \ D_2(b_2) = (\lambda - \nu) b_2 + b_1. \tag{3}$$

As above we can show that $b_1 = \psi_1(\theta)$, $b_2 = \psi_2(\theta)$, where θ is a generator of the subfield $\ker D_1$ and $D_2(\theta) = g(\theta)$ for some rational functions $\psi_1, \psi_2, g \in \mathbb{K}(t)$. Using (3) one can easily show that

$$\psi_1' g = (\lambda - \nu)\psi_1, \ \psi_2' g = (\lambda - \nu)\psi_2 + \psi_1. \tag{4}$$

Since $b_1 \in R\backslash\mathbb{K}$ it holds $\psi_1' \neq 0$. The equality (4) implies the next relations

$$\frac{\psi_1'}{\psi_1} = \frac{(\lambda - \nu)\psi_2'}{(\lambda - \nu)\psi_2 + \psi_1} = \left(\frac{(\lambda - \nu)\psi_2}{\psi_1}\right)' \tag{5}$$

(note that $(\lambda - \nu)\psi_2 + \psi_1 \neq 0$ because ψ_1 and ψ_2 are linearly independent over $\mathbb{K}$). But the relation (5) is impossible if $\lambda \neq \nu$ by Lemma 4. This contradiction shows that $\lambda = \nu$. □

The next statement can be easily deduced from the theorem of S.Lie about solvable Lie algebras.

Lemma 6. *Let V be a finite dimensional vector space over the field $\mathbb{K}$ and T, S be linear operators on V. If $[T, S] = S$, then the operator S is nilpotent.*

2. Finite dimensional solvable subalgebras of $\widetilde{W_2}(\mathbb{K})$

Lemma 7. *Let L be a finite dimensional solvable subalgebra of rank 2 over R of $\widetilde{W_2}(\mathbb{K})$ and let $\langle D_1\rangle$ be its arbitrary one dimensional ideal. Denote $I = RD_1 \cap L$. If the ideal I is abelian, then there exists an element $D_2 \in L\backslash I$ such that L is one of the following algebras:*

(1) $L = \langle D_1, aD_1, \dots, (a^n/n!)D_1, D_2\rangle$, where $a \in R$ such that $D_1(a) = 0, D_2(a) = 1, [D_2, D_1] = \lambda D_1$ and $\lambda = 0$ or $\lambda = 1$, $n \geq 1$. If $n = 0$, we put $L = \langle D_1, D_2\rangle$.

(2) $L = \langle D_1, a_1D_1, \dots, a_nD_1, D_2\rangle$, where $a_i \in R, [D_2, D_1] = D_1, D_1(a_i) = 0, D_2(a_i) = \beta m_i a_i, m_i \in \mathbb{Z}$ for all $i, \beta \in \mathbb{K}^\star, m_i \neq m_j$ for $i \neq j$, $n \geq 1$.

(3) $L = \langle D_1, aD_1, \dots, (a^n/n!)D_1, D_2, bD_1 + aD_2\rangle$, where $a, b \in R$ such that $D_1(a) = 0, D_1(b) = \beta, \beta \in \mathbb{K}$, $[D_2, D_1] = 0,, D_2(a) = 1, D_2(b) = (n+1)\gamma a^n, \gamma \in \mathbb{K}$, $n \geq 1$ (if $n = 0$ we put $L = \langle D_1, D_2, bD_1 + aD_2\rangle$).

Proof. The set $I = RD_1 \cap L$ is an ideal of L by Lemma 3. We can write $I = \langle D_1, a_1D_1, \dots, a_nD_1\rangle$ for some elements $a_i \in R$ and $n \geq 1$ (if $n = 0$ we put $I = \langle D_1\rangle$). Since the ideal I is abelian we have $D_1(a_i) = 0,\ i = 1, \dots, n$. We consider two cases depending on $\dim L/I$ (recall that $\dim L/I \leq 2$ by Lemma 3).

Case 1. $\dim L/I = 1$. Take any element $D_2 \in L\backslash I$. As $\langle D_1\rangle$ is an ideal of L we have $[D_2, D_1] = \nu D_1$ for some $\nu \in \mathbb{K}$. The elements D_1 and D_2 are linearly independent over R by the choice of the ideal I. First, let the linear operator $\operatorname{ad} D_2$ have the only eigenvalue ν on the vector space I (recall that $[D_2, D_1] = \nu D_1$). If $aD_1, bD_1 \in I$ are eigenvectors of $\operatorname{ad} D_2$, i.e. $[D_2, aD_1] = \nu aD_1$, $[D_2, bD_1] = \nu bD_1$, then the elements aD_1, bD_1 are linearly dependent over $\mathbb{K}$ by Lemma 5. Hence D_1 is the unique eigenvector of $\operatorname{ad} D_2$ on I (up to multiplication by a nonzero scalar). But then the linear operator $\operatorname{ad} D_2$ has a Jordan basis in I of the form $\{D_1, a_1D_1, \dots, a_nD_1\}$, $a_i \in R$ such that

$$[D_2, a_iD_1] = \nu a_iD_1 + a_{i-1}D_1, i = 1, \dots, n, [D_2, D_1] = \nu D_1$$

(in case $n = 0$ we have $I = \langle D_1\rangle$). The last relations imply the equalities $D_2(a_i) = a_{i-1}, i = 2, \dots, n$, $D_2(a_1) = 1$. Denoting $a = a_1$ we have $D_2(a_2 - a^2/2!) = 0$ and taking into account the relation $D_1(a_2 - a^2/2!) = 0$ we see by Lemma 1 that $a_2 - a^2/2! \in \mathbb{K}$. But then without loss of generality we can take $a_2 = a^2/2!$. Analogously $D_2(a_3 - a^3/3!) = a_2 - a_2 = 0$ and $D_1(a_3 - a^3/3!) = 0$, so we can put $a_3 = a^3/3!$. Repeating these considerations we get a $\mathbb{K}$-basis $\{D_1, aD_1, \dots, (a^n/n!)D_1\}$ of the ideal I (recall that $I = \langle D_1\rangle$ in case $n = 0$). The algebra Lie L is of type 1 because we always can assume that $\nu = 0$ or $\nu = 1$ choosing a convenient multiple of the element D_2.

Now let $\operatorname{ad} D_2$ have on I at least two different eigenvalues. Our aim is to show that $\operatorname{ad} D_2$ is a diagonalizable operator on I. Denote by $I(\lambda)$ the root space of $\operatorname{ad} D_2$ corresponding to the eigenvalue $\lambda, \lambda \neq \nu$. Since $\operatorname{ad} D_2$ has on $I(\lambda)$ the only eigenvalue λ it follows from the previous considerations that $\operatorname{ad} D_2$ has on $I(\lambda)$ a Jordan basis such that the matrix of $\operatorname{ad} D_2$ in this basis is a single Jordan block. Therefore if $\dim I(\lambda) > 1$ then there exist elements

$aD_1, bD_1 \in I$ such that

$$[D_2, aD_1] = \lambda aD_1, [D_2, bD_1] = \lambda bD_1 + aD_1.$$

The latter is impossible by Lemma 5 and therefore $\dim I(\lambda) = 1$. Choosing any element $D_1' \in I$ with property $[D_2, D_1'] = \lambda D_1'$ instead of the element D_1 and using Lemma 5 we can analogously show that $\dim I(\nu) = 1$, where $I(\nu)$ is the root space corresponding to the eigenvalue ν of $\operatorname{ad} D_2$ on I. Therefore all the root spaces are one-dimensional and $\operatorname{ad} D_2$ is diagonalizable on I.

Since at least one of the eigenvalues of $\operatorname{ad} D_2$ on I is nonzero we can choose elements D_1 and D_2 in such a way that

$$[D_2, D_1] = D_1, I = \langle D_1, a_1D_1, \ldots, a_nD_1\rangle,$$

where $[D_2, a_iD_1] = \lambda_i a_i D_1, \lambda_i \neq \lambda_j$ if $i \neq j$ and $\lambda_i \neq 1, i = 1, \ldots, n$.

Applying Lemma 5 (with $\nu = 1$) we can easily show that $\frac{\lambda_i - 1}{\lambda_1 - 1} = \tau_i \in \mathbb{Q}$, $i = 2, \ldots, n$. Denote $\tau_i = \frac{k_i}{l_i}, k_i, l_i \in \mathbb{Z}, i = 2, \ldots, n$. If l is the least common multiple of $l_2, \ldots, l_n$, then one can write $\tau_i = \frac{m_i}{l}$ and $\lambda_i = m_i\beta + 1$, where $\beta = \frac{\lambda_1 - 1}{l}$ (note that $\lambda_i - 1 = \tau_i(\lambda_1 - 1)$). Thus, L is an algebra of type 2 of Lemma.

Case 2. $\dim L/I = 2$. The quotient algebra L/I is nonabelian by Lemma 3, so it contains a noncentral one-dimensional ideal $\langle D_2 + I\rangle$. Then there exists an element $bD_1 + cD_2 \in L$ such that

$$[bD_1 + cD_2 + I, D_2 + I] = D_2 + I.$$

This means that $[bD_1 + cD_2, D_2] = D_2 + gD_1$ for some element $gD_1 \in I$. Since the ideal I is abelian it is obvious that $\operatorname{ad} D_2 = \operatorname{ad}(D_2 + gD_1)$ on the vector space I over $\mathbb{K}$. We obtain the following relation for linear operators on I :

$$[\operatorname{ad}(bD_1 + cD_2), \operatorname{ad} D_2] = \operatorname{ad}(D_2 + gD_1) = \operatorname{ad} D_2.$$

But then $\operatorname{ad} D_2$ acts nilpotently on I by Lemma 6. In case $\dim I = 1$ we get (after direct calculations) the Lie algebra of type 3 with $n = 0$. Let $\dim I \geq 2$. Since $[D_2, D_1] = 0$ one can easily show (using Lemma 3) that the ideal I can be written in the form $I = \langle D_1, aD_1, \ldots, (a^n/n!)D_1\rangle$ for some $a \in R, D_2(a) = 1, n \geq 1$.

Returning now to the above mentioned element $bD_1 + cD_2 \in L$ we see that

$$[D_1, bD_1 + cD_2] = D_1(b)D_1 + D_1(c)D_2 \in \langle D_1\rangle$$

and therefore $D_1(c) = 0, D_1(b) \in \mathbb{K}$. Further, from the equality

$$[D_2, bD_1 + cD_2] = D_2(b)D_1 + D_2(c)D_2 \in I + \langle D_2\rangle.$$

we obtain $D_2(c) = \gamma \in \mathbb{K}$, $D_2(b) \in \langle 1, a, a^2/2!, \ldots, a^n/n!\rangle$. From the relations $D_2(c) = \gamma \in \mathbb{K}$ and $D_2(a) = 1$ it follows that $D_2(\gamma a - c) = 0$. Then Lemma 1 yields $\gamma a - c \in \mathbb{K}$, i.e. $c = \gamma a + b$ for some $\gamma, \beta \in \mathbb{K}$.

The element $D_3 = \gamma^{-1}(bD_1 + cD_2 - \beta D_2)$ of the algebra L can be written in the form $D_3 = b_1D_1 + aD_2$ for some $b_1 \in R$. As $D_2(b_1) \in \langle 1, a, a^2/2!, \ldots, a^n/n!\rangle$ we can substract from $b_1D_1 + aD_2$ a suitable linear combination of the elements $D_1, aD_1, a^2/2!D_1, \ldots, a^n/n!D_1$ and assume without loss of generality that $D_2(b_1) = (n+1)\gamma a^n$ for some $\gamma \in \mathbb{K}$. Denoting $b = b_1$, $\beta = D_1(b) \in \mathbb{K}$ we see that L is of type 3 of this Lemma.

$\square$

Remark 2. For each type of Lie algebras from Lemma 7 one can easily point out a realization:

1. $\lambda = 0, D_1 = \frac{\partial}{\partial x}, D_2 = \frac{\partial}{\partial y}, a = y$. $\lambda = 1, D_1 = \frac{\partial}{\partial x}, D_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial x}, a = y$.
2. $D_1 = \frac{\partial}{\partial x}, a_i = y^{m_i}, D_2 = \beta y\frac{\partial}{\partial y} - x\frac{\partial}{\partial x}, \beta \in \mathbb{K}$.
3. $D_1 = \frac{\partial}{\partial x}, D_2 = \frac{\partial}{\partial y}$, $a = y, b = \beta x + \gamma y^{n+1}, \beta, \gamma \in \mathbb{K}$.

Lemma 8. *Let L be a subalgebra of $\widetilde{W_2}(\mathbb{K})$ satisfying all the conditions of the previous Lemma with the exception of that the ideal I is abelian. If I is nonabelian, then there exist elements $D_1 \in I, D_2 \in L\backslash I$ such that L is one of the following algebras:*

(1) $L = \langle D_1, aD_1, \ldots, (a^{n-1}/(n-1)!)D_1, bD_1, D_2\rangle$, where $a, b \in R$ such that $D_1(a) = 0, D_2(a) = 1, D_1(b) = -1, D_2(b) = 0, [D_2, D_1] = 0$.

(2) $L = \langle D_1, a_1D_1, \ldots, a_{n-1}D_1, bD_1, D_2\rangle$, where $a_i, b \in R$ such that $[D_2, D_1] = D_1, D_1(a_i) = 0, D_1(b) = -1, D_2(b) = -b, D_2(a_i) = \beta m_i a_i$ for some $m_i \in \mathbb{Z}, \beta \in \mathbb{K}^\star$ and $m_i \neq m_j$ if $i \neq j$.

(3) $L = \langle D_1, aD_1, \ldots, (a^{n-1}/(n-1)!)D_1, (v-\alpha a^n)D_1, D_2, (-\beta v+\gamma(a^n/n!))D_1-aD_2\rangle$, where $a, v \in R$ such that $[D_1, D_2] = 0, D_1(a) = 0, D_2(a) = 1, D_1(v) = -1, D_2(v) = 0, \alpha, \beta \in \mathbb{K}$, and $\gamma = \alpha(\beta - n)$.

Proof. Let $\langle D_1\rangle$ be the one-dimensional ideal of L lying in I. The ideal I has by Lemma 2 a basis over $\mathbb{K}$ of the form $\{D_1, a_1D_1, \ldots, a_{n-1}D_1, bD_1\}$, where $D_1(a_i) = 0, D_1(b) = -1, i = 1, \ldots, n-1$ (for $n = 0$ we put $I = \langle D_1, bD_1\rangle$ with $D_1(b) = -1$). Suppose that $n = 0$, i.e. $\dim I = 2$. If $\dim L/I = 1$, then $L = \langle D_1, bD_1, D_2\rangle$ is of type 2 or 3. If $\dim L/I = 2$, then L/I is nonabelian by Lemma 3 and taking into account that L/I is nonabelian we have $L = I \oplus J$ for nonabelian ideal J of dimension 2. Then L is of type 3. So we may assume that $\dim I \geq 3$. As in the previous Lemma we divide the proof into following cases:

Case 1. $\dim L/I = 1$. Take any element $D_2 \in L\backslash I$. Then $[D_2, bD_1] = \lambda bD_1 + cD_1$, where $cD_1 \in I' = [I, I]$ because $\dim L/I' = 2$ and $\langle bD_1 + I'\rangle$ is a one-dimensional ideal of L/I'. If $\lambda \neq 0$, then we may assume without loss of generality that $\lambda = 1$, and then

$$[\operatorname{ad} D_2, \operatorname{ad}(bD_1)] = \operatorname{ad}(bD_1 + cD_1) = \operatorname{ad}(bD_1)$$

on I' because I' is an abelian ideal of L. But then the linear operator $\operatorname{ad}(bD_1)$ acts nilpotently on I' by Lemma 6. The latter is impossible and therefore $\lambda = 0$. This means that L/I' is an abelian Lie algebra of dimension 2. As $[D_2, bD_1] = cD_1$ for some element $cD_1 \in I'$ we get $[D_2 + cD_1, bD_1] = 0$ (recall that $[bD_1, cD_1] = cD_1$ for all $cD_1 \in I'$). So, we can choose the element D_2 in such a way that $[D_2, bD_1] = 0$. If the linear operator $\operatorname{ad} D_2$ has on $I' = \langle D_1, \ldots, a_{n-1}D_1\rangle$ at least two different eigenvalues, then there exists by Lemma 5 a basis $\{D_1, \ldots, a_{n-1}D_1\}$ of I' such that $D_2(a_i) = m_i\beta a_i$, for some $m_i \in \mathbb{Z}, \beta \in \mathbb{K}^\star$, $m_i \neq m_j$ if $i \neq j, [D_2, D_1] = D_1$. Then from the relation $[D_2, bD_1] = 0$ it follows $D_2(b) = -b$. The algebra L is of type 2 of Lemma.

Now let $\operatorname{ad} D_2$ have the only eigenvalue μ on I'. If $\mu = 0$, then L is obviously the Lie algebra of type 1 of Lemma. Let $\mu \neq 0$. Taking a suitable multiple of D_2 we may assume that $\mu = 1$. Then replacing the element D_2 by the element $D_2 - bD_1$ we get the case $\mu = 0$ and L is again of type 1 of Lemma.

Case 2. $\dim L/I = 2$. As in the case 1 take a one-dimensional ideal $\langle D_1\rangle$ of L lying in I' and a basis of I of the form $\{D_1, a_1D_1, \ldots a_{n-1}D_1, bD_1\}$ such that $D_1(a_i) = 0$, $D_1(b) = -1$, $i = 0, \ldots n-1$. Let $\langle D_2 + I\rangle$ be the one-dimensional ideal of the nonabelian quotient algebra L/I. Accordingly to Case 1 the algebra $\langle D_2\rangle + I$ is of type 1 or type 2 of this Lemma. Let us show that $\langle D_2\rangle + I$ is of type 1 of this Lemma, i. e. the linear operator $\operatorname{ad} D_2$ acts

nilpotently on I'. Really since $\langle bD_1 + I'\rangle$ is an ideal of the algebra L/I' and $\mathrm{ad}(bD_1)$ acts on I' as the identity operator the ideal $\langle bD_1 + I'\rangle$ lies in the center of L/I' (because of Lemma 6), i. e. $[D, bD_1] \in I'$ for any element $D \in L$. Take any element $cD_1 + dD_2 \in L\backslash I$ such that $[cD_1 + dD_2, D_2] = D_2 + rD_1$ for some element $rD_1 \in I$. The element rD_1 can be written in the form $rD_1 = \mu bD_1 + r_1D_1$, where $\mu \in \mathbb{K}$, $r_1D_1 \in I'$. But then we obtain

$$[cD_1 + bD_2, D_2 + \mu bD_1] = (D_2 + \mu bD_1) + r_2D_1$$

for some element $r_2D_1 \in I'$ The latter means that $\mathrm{ad}(D_2 + \mu bD_1)$ acts nilpotently on I' (by Lemma 6). Replacing the element D_2 by the element $D_2 + \mu bD_1$ we can assume without loss of generality that $\mathrm{ad}\, D_2$ is a nilpotent linear operator on I'. So, the subalgebra $\langle D_2\rangle + I$ is of type 1 of this Lemma and hence $I' + \langle D_2\rangle$ can be written in the form

$$I' + \langle D_2\rangle = \langle D_1, aD_1, \dots \frac{a^{n-1}}{(n-1)!}D_1, D_2, \rangle$$

where $[D_2, D_1] = 0$, $D_1(a) = 0$, $D_2(a) = 1$.

Further, it follows from the above mentioned equality

$$[cD_1 + dD_2, D_2] = D_2 + r_2D_1 \tag{6}$$

that $D_2(d) = -1$. Analogously we obtain $D_1(d) = 0$, $D_1(c) = \beta_1 \in \mathbb{K}$ from the relation $[cD_1 + dD_2, D_1] \in \langle D_1\rangle$. Since $D_2(a) = 1$ and $D_2(d) = -1$ we have $D_2(a + d) = 0$. Taking into account the equality $D_1(a + d) = 0$ we obtain by Lemma 1 that $a + d = \alpha_1 \in \mathbb{K}$. But then $d = -a + \alpha_1$ and without loss of generality we can choose $cD_1 - aD_2$ instead of the element $cD_1 + dD_2$.

Since $[D_2, bD_1] \in I'$ (as we have proved before) we see that

$$D_2(b) = \alpha_0 + \alpha_1 a + \cdots + \alpha_{n-1}\frac{a^{n-1}}{(n-1)!}$$

for some $\alpha_i \in \mathbb{K}$. Put $v = b - \alpha_0 a - \alpha_1\frac{a^2}{2!} - \cdots - \alpha_{n-1}\frac{a^n}{n!}$. Then $D_1(v) = D_1(b) = -1, D_2(v) = 0$. Substracting the element $(\alpha_0 a + \alpha_1\frac{a^2}{2!} + \cdots + \alpha_{n-2}\frac{a^{n-1}}{(n-1)!})D_1 \in I'$ from the element bD_1 we can assume without loss of generality that $b = v - \alpha_{n-1}\frac{a^n}{n!}$ for some $\alpha_{n-1} \in \mathbb{K}$. Then $D_1(b) = -1$, $D_2(b) = \alpha_{n-1}\frac{a^{n-1}}{(n-1)!}$. Further, recall that for the basic element $cD_1 - aD_2$ we have $D_1(c) = \beta_1 \in \mathbb{K}$.

Rewriting the relation 6 in the form $[cD_1 - aD_2, D_2] = D_2 + r_2D_1$ we obtain that

$$D_2(c) = \gamma_0 + \gamma_1 a + \cdots + \gamma_{n-1}\frac{a^{n-1}}{(n-1)!} \text{ for some } \gamma_i \in \mathbb{K},\ i = 1, \dots, n-1.$$

Substracting the element $(\gamma_0 a + \gamma_1\frac{a^2}{2!} + \cdots + \gamma_{n-2}\frac{a^{n-1}}{(n-1)!})D_1 \in I'$ from the element $cD_1 - aD_2$ we may assume without loss of generality that $D_2(c) = \gamma_{n-1}\frac{a^{n-1}}{(n-1)!}$. Suppose that $\beta_1 = D_1(c) \neq 0$. Since $D_1(\beta_1^{-1}c + v - \beta_1^{-1}\gamma_{n-1}\frac{a^n}{n!}) = 0$ and $D_2(\beta_1^{-1}c + v - \beta_1^{-1}\gamma_{n-1}\frac{a^n}{n!}) = \beta_1^{-1}\gamma_{n-1}\frac{a^{n-1}}{(n-1)!} - \beta_1^{-1}\gamma_{n-1}\frac{a^{n-1}}{(n-1)!} = 0$ we have by Lemma 1 that $\beta_1^{-1}c + v - \beta_1^{-1}\gamma_{n-1}\frac{a^n}{n!} = \nu$ for some $\nu \in \mathbb{K}$. Substracting the element $\nu D_1 \in I'$ from the element $cD_1 + aD_2$ we may assume that $\nu = 0$. Then we obtain $c = -\beta_1 v + \gamma_{n-1}\frac{a^n}{n!}$. Denoting α_{n-1} by α, γ_{n-1} by γ and β_1 by β we obtain

a basis of L of the form:

$$\{D_1, aD_1, \dots, \frac{a^{n-1}}{(n-1)!}D_1, (v - \alpha\frac{a^n}{n!})D_1, D_2, (-\beta v + \gamma\frac{a^n}{n!})D_1 - aD_2)\}$$

(here $D_1(a) = 0, D_1(v) = -1, D_2(a) = 1, D_2(v) = 0$). Now suppose that $\beta = D_1(c) = 0$. Since $D_2(c) = \gamma\frac{a^{n-1}}{(n-1)!}$ we see that for the element $c_1 = c - \gamma\frac{a^n}{n!}$ it holds $D_1(c) = \beta = 0$, $D_2(c) = 0$. So by Lemma 1 we obtain $c - \gamma\frac{a^n}{n!} = \nu_2$ for some $\nu_2 \in \mathbb{K}$. Substracting the element $\nu_2 D_1$ from $cD_1 + aD_2$ we may assume that $\nu_2 = 0$. So we have that $c = \gamma\frac{a^n}{n!}$ i.e. the basis of L is of the same form as in case $\beta \neq 0$.

Now consider the product $[(v - \alpha a^n/n1)D_1, (\beta v + \gamma a^n/n!)D_1 - aD_2]$. This product equals to $(-\alpha\beta + \gamma + n\alpha)D_1$ and belongs to I'. Hence $-\alpha\beta + \gamma + n\alpha = 0$ and $\gamma = \alpha(\beta - n)$. We see that L is of type 3 of Lemma. □

Remark 3. There exist realizations for all types of Lie algebras from Lemma 8:
1. $D_1 = \frac{\partial}{\partial x}, D_2 = \frac{\partial}{\partial y}$, $a = y, b = -x$
2. $D_1 = \frac{\partial}{\partial x}, D_2 = \beta y\frac{\partial}{\partial y} - x\frac{\partial}{\partial x}$, $a = y, b = -x, a_i = y^{m_i}, .$
3. $D_1 = \frac{\partial}{\partial x}, D_2 = \frac{\partial}{\partial y}$, $a = y, f = -x$

The next three corollaries can be easily proved by using results of Lemmas 2, 7 and 8.

Corollary 1. *Let L be a finite dimensional nilpotent subalgebra of $\widetilde{W}_2(\mathbb{K})$. Then there exist elements $D_1, D_2 \in L$ linearly independent over R such that L is one of the following algebras:*
(1) $L = \langle D_1, a_1D_1, \dots, a_nD_1\rangle$, *for some $a_i \in R$ such that $D_1(a_i) = 0, i = 1, \dots, n$.*
(2) $L = \langle D_1, D_2\rangle, [D_1, D_2] = 0$.
(3) $L = \langle D_1, aD_1, \dots, (a^n/n!)D_1, D_2\rangle$ *for some $a \in R$ such that $D_1(a) = 0, D_2(a) = 1$,* $[D_1, D_2] = 0$.

Corollary 2. *Let L be a finite dimensional solvable subalgebra of $\widetilde{W}_2(\mathbb{K})$. If L is nonabelian and decomposable into a direct sum of proper ideals, then $L = A \oplus B$, where A is a nonabelian ideal of dimension 2 and B is either a one-dimensional ideal or a two-dimensional nonabelian ideal of L.*

Corollary 3. *Let L be a finite dimensional solvable subalgebra of $\widetilde{W}_2(\mathbb{K})$. If L is nonabelian, then* $\dim L/L' \leq 2$.

3. Nonsolvable subalgebras of $\widetilde{W}_2(\mathbb{K})$

Lemma 9. *If L is a finite dimensional semisimple subalgebra of the Lie algebra $\widetilde{W}_2(\mathbb{K})$, then L is isomorphic to $sl_2(\mathbb{K})$ or $sl_3(\mathbb{K})$, or $sl_2(\mathbb{K}) \oplus sl_2(\mathbb{K})$.*

Proof. If L is of rank 1 (as a system of vectors) over R, then $L \simeq sl_2(\mathbb{K})$ by Lemma 2. So, we can assume that L is of rank 2 over R. Fix a Cartan subalgebra H of the algebra L, a basis π of the system Δ of roots which correspond to H and let Δ^+ be the set of positive roots relatively to the ordering on Δ. Consider the triangular decomposition

$$L = N_+ + H + N_-,\ N_+ = \oplus_{\alpha_i > 0} L_{\alpha_i},\ N_- = \oplus_{\alpha_i < 0} L_{\alpha_i}$$

and the Borel subalgebra $B = H + N_+$ of L. If the subalgebra N_+ is abelian, then L is a direct sum $L = L_1 \oplus \dots \oplus L_k$ of ideals isomorphic to $sl_2(\mathbb{K})$ (see, for example [6]). Then B

is a direct sum $B = B_1 \oplus \cdots \oplus B_k$ of Borel subalgebras of $L_i \simeq sl_2(\mathbb{K})$ and using Corollary 2 we see that either $L = L_1 \simeq sl_2(\mathbb{K})$ or $L = L_1 \oplus L_2 \simeq sl_2(\mathbb{K}) \oplus sl_2(\mathbb{K})$.

Now, let the subalgebra N_+ be nonabelian. Since N_+ is nilpotent it is indecomposable into a direct sum of nonzero ideals by Corollary 1. But then the algebra L is also indecomposable into a direct sum of proper ideals and hence is simple. By Corollary 3 we have relations:

$$\dim B/B' = \dim B/N = \dim H \leq 2.$$

Therefore, if N_+ is nonabelian, then $\dim H = 2$ and L is a simple Lie algebra of one of the types A_2, B_2 or G_2. First suppose that L is of type G_2. Then the subalgebra N_+ from its triangular decomposition has nonabelian derived subalgebra $[N_+, N_+]$. The latter is impossible (see Corollary 1) and hence L cannot be of type G_2.

Further, let us show that L is not of type B_2. Fix a Cartan subalgebra H of L and a basis $\{\alpha, \beta\}$ of the root system Δ. Then the subalgebra N_+ has the basis $\{e_\alpha,\ e_\beta,\ e_{\alpha+\beta},\ e_{\alpha+2\beta}\}$. It follows from Corollary 1 that $e_{\alpha+\beta} = f \cdot e_{\alpha+2\beta}$ for some element $f \in R$. Consider the element σ_α of the Weyl group of the root system Δ acting by the rule $\sigma_\alpha(\gamma) = \gamma - \frac{2(\gamma,\alpha)}{(\alpha,\alpha)}\alpha$, where γ is an arbitrary root from Δ. Then $\{-\alpha, \beta + \alpha.\beta, \alpha + 2\beta\}$ are positive roots relatively to the new basis $\{\sigma_\alpha(\alpha), \sigma_\alpha(\beta)\}$. The subalgebra $\langle e_{-\alpha}, e_{\beta+\alpha}, e_\beta, e_{\alpha+2\beta}\rangle$ is nilpotent and by Corollary 1 it holds $e_\beta = g \cdot e_{\alpha+2\beta}$ for some $g \in R$. Analogously one can show that $e_\alpha = h \cdot e_{\alpha+2\beta}$ for some $h \in R$. Three relations with coefficients f, g, h obtained above imply that all elements from the basis of N_+ are multiple to one of them and hence the subalgebra N_+ is abelian by Lemma 2. This is impossible and obtained contradiction shows that L is not of type B_2. Thus, L is of type A_2. □

Lemma 10. *Let L be a finite dimensional nonsolvable subalgebra of $\widetilde{W_2}(\mathbb{K})$ whose Levi factor is either of type A_2 or of type $A_1 \times A_1$. Then L is semisimple of type A_2 or of type $A_1 \times A_1$ respectively.*

Proof. Let $S = S(L)$ be the solvable radical of L. By Theorem of Levi-Maltsev $L = L_1 \leftthreetimes S$, where L_1 is a Levi factor of L. First suppose that L_1 is of type A_2. Let us fix a Cartan subalgebra H of L_1 and the root system Δ corresponding to H. Consider the triangular decomposition

$$L = N_- + H + N_+ \tag{7}$$

of L_1 relatively to H and Δ. Since the subalgebra N_+ is nonabelian (this follows from the multiplication law in algebras of type A_2) it contains by Corollary 1 elements D_1 and D_2, linearly independent over R such that $[D_1, D_2] = 0$. Consider S as an L_1-module and take the older vector $D \in S$ relatively to the decompostion (7). Then we have

$$[D_1, D] = 0, \quad [D_2, D] = 0. \tag{8}$$

If we write $D = aD_1 + bD_2$ for some $a, b \in R$, then from the previous relation we get

$$D_1(a) = 0, D_1(b) = 0, D_2(a) = 0 \ \text{ and } \ D_2(b) = 0.$$

Lemma 1 yields now that $a, b \in \mathbb{K}$, i.e. $D \in L_1$. As $L_1 \cap S = 0$ we obtain $S = 0$ and therefore $L = L_1$ is a simple Lie algebra of type A_2.

Let now L_1 be of type $A_1 \times A_1$. Write $L_1 = G_1 \oplus G_2$, where $G_i \simeq sl_2(\mathbb{K})$ and fix Cartan subalgebras $H_1 \subset G_1, H_2 \subset G_2$. Consider any triangular decompositions

$$G_1 = N_{1+} + H_1 + N_{1-},\ G_2 = N_{2+} + H_2 + N_{2-}$$

relatively to H_1 and H_2. Take any nonzero element $D_1 \in N_{1+}$. Then at least one of the subalgebras N_{1-}, N_{2+}, N_{2-} contains a nonzero element D_2 such that D_1 and D_2 are linearly independent over R. Really, in other case $H_1 = [N_{1+}, N_{1-}]$ and $H_2 = [N_{2+}, N_{2-}]$ lie also in RD_1 and therefore $L = G_1 \oplus G_2 \subset RD_1$ which is impossible by Lemma 2. It is easily shown that the two-dimensional abelian subalgebra $N_+ = \langle D_1, D_2\rangle$ is a part of triangular decomposition $L = N_+ + H + N_-$ of L relatively to the Cartan subalgebra $H = H_1 \oplus H_2$. Choosing as above the older vector in S relatively to N_+ and repeating the considerations from the case $L_1 \simeq A_2$ we get $S = 0$, i.e. L is semisimple of type $A_1 \times A_1$. □

Lemma 11. *Let L be a nonsolvable finite dimensional subalgebra of $\widetilde{W}_2(\mathbb{K})$. Then L is isomorphic to one of the following algebras:*

(1) $sl_3(\mathbb{K})$.

(2) $sl_2(\mathbb{K})$ *or* $sl_2(\mathbb{K}) \oplus sl_2(\mathbb{K})$.

(3) $sl_2(\mathbb{K}) \leftthreetimes V_m$, *where V_m is the irreducible module over $sl_2(\mathbb{K})$ of dimension $m+1$, $m = 0, 1, \ldots$.*

(4) $gl_2(\mathbb{K}) \leftthreetimes V_m$, *where V_m is the irreducible module over $gl_2(\mathbb{K})$ of dimension $m+1$, $m = 0, 1, \ldots$ and nonzero central elements of $gl_2(\mathbb{K})$ act on V_m as nonzero scalars.*

Proof. Let S be the solvable radical of L and L_1 be a Levi factor of the algebra L. We can consider only the case $S \neq 0$ because of Lemma 9. It follows from Lemma 10 that $L_1 \simeq sl_2(\mathbb{K})$. Choose a Cartan subalgebra H of the algebra L_1 and a triangular decomposition $L_1 = N_+ + H + N_-$ of L_1.

Case 1. $\dim S = 1$ or $\dim S = 2$. If $\dim S = 1$, then $L = L_1 \oplus S$ is a sum of two ideals and $L \simeq sl_2(\mathbb{K}) \oplus V_0$, where V_0 is a one-dimensional module over $sl_2(\mathbb{K})$. The algebra L is of type 4 with $m = 0$. Suppose that $\dim S = 2$. If S is a nonabelian ideal of L, then L is a direct sum of ideals $L = L_1 \oplus S$. Since $S = \langle w\rangle \leftthreetimes \langle v_0\rangle$ for some elements $w, v_0 \in S$, then $L \simeq gl_2(\mathbb{K}) \leftthreetimes \langle v_0\rangle$ is of type (5) with $m = 0$ because $L_1 \oplus \langle w\rangle \simeq gl_2(\mathbb{K})$. Let S be abelian. Suppose that S is a reducible module. Then $S = S_1 \oplus S_2$ is a direct sum of L_1-modules of dimension 1 over $\mathbb{K}$. Take the Borel subalgebra $B = H + N_+$ of L_1. Then the subalgebra $B \oplus S_1 \oplus S_2$ of L is solvable of dimension 4. But such an algebra does not exist by Lemmas 7 and 8. This contradiction shows that S is irreducible and $L \simeq sl_2(\mathbb{K}) \leftthreetimes V_1$, where V_1 is of dimension 2 over $\mathbb{K}$. The algebra L is of type 4. Further, we will assume that $\dim S \geq 3$.

Case 2. S is abelian (of dimension ≥ 3). Let us show that S is an irreducible module over L_1. Assume to the contrary that S is reducible. If S is a sum of one-dimensional submodules over L_1, then $L = L_1 \oplus S$ is a direct sum of ideals. Its subalgebra $B + S$ is solvable, nonabelian and decomposable into direct sum of subalgebras $B \oplus S$. The latter is impossible by Corollary 2. So we can assume $S = S_1 \oplus S_2$ where S_1, S_2 are L_1-submodules, $\dim S_1 \geq 2$ and S_1 is irreducible (note that S_1 and S_2 are ideals of L because S is abelian). Let $D_2 \in N_+$ be a nonzero element. Then the subalgebra $M = \langle D_2\rangle + S$ is nonabelian, nilpotent and $\dim M/[M, M] \leq 2$ by Corollary 3. On the other hand, since $[M, M] = [D_2, S_1] \oplus [D_2, S_2]$, $\dim S_i/[D_2, S_i] \geq 1$, $i = 1, 2$ (because $\operatorname{ad} D_2$ acts nilpotently on S_i) we have

$$\dim M/[M, M] = \dim\langle D_2\rangle + \dim S_1/[D_2, S_1] + \dim S_2/[D_2, S_2] \geq 3.$$

The latter contadicts to Corollary 3 and hence S is a simple L_1-module. It is obvious that L is of type 4. Note that the subalgebra $M = \langle D_2\rangle + S$ is of the form

$$\langle D_2, D_1, aD_1, \dots \frac{a^k}{k!}D_1\rangle,\ [D_2, D_1] = 0,\ D_1(a) = 0,\ D_2(a) = 1.$$

Case 3. S is a nilpotent (nonabelian) ideal. Then by Corollary 1 there exist elements $D_1, D_2 \in S$ such that

$$S = \langle D_2, D_1, aD_1, \dots, (a^k/k!)D_1\rangle,\ [D_2, D_1] = 0,\ D_1(a) = 0,\ D_2(a) = 1,\ \dim S \geq 3.$$

Therefore $\langle D_1\rangle = S^{k-1}$ and $\langle D_1\rangle$ is an ideal of L. Using Lemma 3 we see that $RD_1 \cap L$ is an ideal of L and therefore $L_1 \leftthreetimes \langle D_1, aD_1, \dots \frac{a^k}{k!}D_1\rangle$ is a subalgebra of L. This subalgebra has the abelian decomposable ideal $\langle D_1, aD_1, \dots \frac{a^k}{k!}D_1\rangle$. This is impossible by the Case 1 and therefore the Case 3 is impossible.

Case 4. S is solvable (nonnilpotent). The L_1-submodule $S' = [S, S]$ is nilpotent, therefore S' is abelian by the previous case and S' is an irreducible L_1-module by Cases 1 and 2. Since $\dim S/S' \leq 2$ by Corollary 3 we have a direct decomposition $S = S' \oplus S_2$ of L_1-submodules with $\dim S_2 \leq 2$. First suppose that $\dim S_2 = 2$. Let us show that S_2 is an irreducible L_1-module. Indeed, in other case $S_2 \subseteq C_S(L_1)$ and the centralizer $C_S(L_1)$ a submodule of the L-module S. Because of previous cases we can assume that $\dim S' \geq 2$ and hence S' is an irreducible L_1-module. Then obviously $C_S(L_1) = S_2$. Since $C_S(L_1) = S_2$ is a subalgebra of L the sum $S_2 + L_1$ is a subalgebra of L. The latter is impossible because the subalgebra $S_2 + L_1$ does not exist by the Case 1. This contradiction shows that S_2 is an irreducible L_1-module.

Choose any nonzero elements $D_2 \in N_+$ and $h \in H$ and take standard bases $\{e_0, e_1\} \subset S_2$ and $\{f_0, f_1, \dots, f_m\} \subset S'$ of the L_1-modules S_2 and S' respectively (recall that $L_1 \simeq sl_2(\mathbb{K})$). Then the linear operator $\operatorname{ad} h$ has eigenvalues $1, -1$ on S_2. If the eigenvalues of $\operatorname{ad} h$ on S' are even, then the elements $[e_i, f_j]$ are eigenvectors for $\operatorname{ad} h$ with odd eigenvalues. Since $[e_i, f_j] \in S'$ we see that $[e_i, f_j] = 0$. Let now the eigenvalues of $\operatorname{ad} h$ on S' be odd. Then $[e_i, f_j]$ are eigenvectors for $\operatorname{ad} h$ with even eigenvalues, so $[e_i, f_j] = 0$, $i = 0, 1$, $j = 0, 1, \dots m$. As S' is abelian the latter means that $S' \subset Z(S)$. This is impossible because of our assumption on S and therefore $\dim S/S' = 1$. Hence $\dim S_2 = 1$. The subalgebra $S_2 + L_1$ is obviously isomorphic to $gl_2(\mathbb{K})$ and S' is an irreducible $S_2 + L_1$-module. Since S_2 lies in the center of $S_2 + L_1$ and S is nonabelian we see that each nonzero element of S_2 acts on S' as multiplication by a nonzero scalar. We get a Lie algebra of type 5 from this Lemma.

□

Remark 4. For each type of Lie algebras from this Lemma one can easily point out its realization:

(1) $\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial x}, x\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}, x(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}), y(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})\rangle \simeq sl_3(\mathbb{K})$;

(2) $\langle \frac{\partial}{\partial x}, -x^2\frac{\partial}{\partial x}, -2x\frac{\partial}{\partial x}\rangle \simeq sl_2(\mathbb{K})$ and $\langle \frac{\partial}{\partial x}, -x^2\frac{\partial}{\partial x} - 2x\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, -y^2\frac{\partial}{\partial y}, -2y\frac{\partial}{\partial y}\rangle \simeq sl_2(\mathbb{K}) \oplus sl_2(\mathbb{K})$;

(3) $\langle x\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}, x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, x^m(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}), x^{m-1}y(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}), \dots, y^m(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})\rangle \simeq sl_2(\mathbb{K}) \leftthreetimes V_m$.

(4) $\langle x\frac{\partial}{\partial x}, x\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}, x^m(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}), x^{m-1}y(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}), \dots, y^m(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})\rangle \simeq gl_2(\mathbb{K}) \leftthreetimes V_m$.

We give a description of finite dimensional subalgebras of the Lie algebra $\widetilde{W}_2(\mathbb{K})$ up to isomorphism as Lie algebras. In fact we give more information about such Lie algebras (up to choice of basis $\{D_1, D_2\}$ of the two-dimensional vector space $\widetilde{W}_2(\mathbb{K})$ over the field

$R = \mathbb{K}(x, y)$). In order to clarify the structure of described subalgebras of $\widetilde{W}_2(\mathbb{K})$ we formulate the main Theorem in terms of generators and relations.

Theorem 1. *Let L be a nonzero finite dimensional subalgebra of the Lie algebra $\widetilde{W}_2(\mathbb{K})$. Then the algebra L belongs to one of the following types:*

(1) $L = \langle e_1, \ldots, e_n \rangle$, *where* $[e_i, e_j] = 0$, $i, j = 1 \ldots n$.

(2) $L = \langle e_1, \ldots, e_n, f \rangle$, *where* $[e_i, e_j] = 0, [f, e_i] = e_i$, $i = 1 \ldots n$.

(3) $L = \langle e_0, \ldots, e_n, f \rangle$, *where* $[e_i, e_j] = 0$, $i, j = 0 \ldots n$, $[f, e_0] = \lambda e_0$, $[f, e_i] = \lambda e_i + e_{i-1}$, $i = 1 \ldots n$, $\lambda = 0$ *or* $\lambda = 1$.

(4) $L = \langle e_0, \ldots, e_n, f \rangle$, *where* $[e_i, e_j] = 0$, $i, j = 0 \ldots n$, $[f, e_i] = (1 + \beta m_i) e_i$, $i = 0 \ldots n$, $m_i \in \mathbb{Z}$, $\beta \in \mathbb{K}^\star$ *and* $m_i \neq m_j$ *provided that* $i \neq j$.

(5) $L = \langle e_0, \ldots, e_n, f, g \rangle$, *where* $[e_i, e_j] = 0$, $i, j = 0 \ldots n$, $[f, e_0] = 0$, $[f, e_i] = e_{i-1}$, $i = 1 \ldots n$, $[g, e_i] = (i - \beta) e_i$, $i = 0 \ldots n$, $[g, f] = f - \gamma e_n$, $\beta, \gamma \in \mathbb{K}$.

(6) $L = \langle e_0, \ldots, e_n, f, g \rangle$, *where* $[e_i, e_j] = 0$, $i, j = 0 \ldots n$, $[f, e_i] = e_i$, $i = 0 \ldots n$, $[g, e_0] = 0$, $[g, e_i] = e_{i-1}$, $i = 1 \ldots n$, $[f, g] = 0$.

(7) $L = \langle e_0, \ldots, e_n, f, g \rangle$, *where* $[e_i, e_j] = 0$, $i, j = 0 \ldots n$, $[f, e_i] = e_i$, $i = 0 \ldots n$, $[g, e_i] = (1 + \beta m_i) e_i$, $i = 0 \ldots n$, $[g, f] = 0$, $\beta \in \mathbb{K}^\star$, $m_i \in \mathbb{Z}$, *and* $m_i \neq m_j$ *if* $i \neq j$.

(8) $L = \langle e_0, \ldots, e_n, f, g, h \rangle$, *where* $[e_i, e_j] = 0$, $i, j = 0 \ldots n$, $[f, e_0] = 0$, $[f, e_i] = e_{i-1}$, $i = 1 \ldots n$, $[g, e_i] = e_i$, $i = 0 \ldots n$, $[g, f] = \alpha e_n$, $[h, e_i] = -(\beta + i) e_i$, $[h, f] = f - \gamma e_n$, $[h, g] = 0, \alpha, \beta \in \mathbb{K}, \gamma = \alpha(\beta - n)$.

(9) $L \simeq sl_2(\mathbb{K})$, *or* $L \simeq sl_2(\mathbb{K}) \oplus sl_2(\mathbb{K})$*;*

(10) $L \simeq sl_3(\mathbb{K})$*;*

(11) $sl_2(\mathbb{K}) \leftthreetimes V_m$, *where* V_m *is the irreducible module over* $sl_2(\mathbb{K})$ *of dimension* $m + 1$, $m = 0, 1, \ldots$*;*

(12) $gl_2(\mathbb{K}) \leftthreetimes V_m$, *where* V_m *is the irreducible module over* $gl_2(\mathbb{K})$ *of dimension* $m + 1$, $m = 0, 1, \ldots$ *and nonzero central elements of* $gl_2(\mathbb{K})$ *act on* V_m *as nonzero scalars.*

Proof. Let L be a finite dimensional solvable subalgebra of the Lie algebra $\widetilde{W}_2(\mathbb{K})$. If L is of rank 1 over R, then L is of type 1 or 2 by Lemma 2. Let L be of rank 2 over R. If L possesses an abelian ideal I of rank 1 over R which is maximal with this property, then L is of type 3, 4 or 5 by Lemma 7 (we denote $e_i = a_i D_1$ in type 4 and $e_i = (a^i/i!) D_1$ for types 3 and 5). Let the ideal I be nonabelian. Then by Lemma 8 L is one of types 6, 7 or 8 (as above we denote $e_i = a_i D_1$ in type 7 and $e_i = (a^i/i!) D_1$ for types 6 and 8, $f = bD_1$ for types 6 and 7 and $f = D_2, g = (v - \alpha(a^n/n!)) D_1$ for type 8 of this Theorem). Further, let L be nonsolvable. If L is semisimple, then L is one of types 9 or 10 by Lemma 9. Finally, if solvable radical of L is nonzero, then L is either of type 11 or of type 12 by Lemma 11.

□

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